Regularity of σ-finite Radon measures.

(on LCH space X)

standing assumption until further notice.

Thm 1. Every σ-finite Radon measure is regular.

Pf. By def., every Radon measure is outer reg. on $\mathcal{B}_X$, so just need to verify inner reg. on $\mathcal{B}_X$. We have the following lemma, which will complete the pf. $\blacksquare$

Lemma 1. A Radon measure μ is inner reg. on every σ-finite set $E \in \mathcal{B}_X$.

Pf of L1. We shall only do the case $\mu(E) < \infty$ and leave the general σ-finite case to the reader (or see Folland; think about how σ-finite is used.)

Pick $\varepsilon > 0$. By outer reg., $\exists$ open $U \supseteq E$ s.t.

$$\mu(E) > \mu(U) - \varepsilon.$$

By inner reg. on open sets, $\exists$ cpct $K \subseteq U$ s.t. $\mu(K) > \mu(U) - \varepsilon \quad \Rightarrow$

$$\mu(U \setminus E) \le \cancel{\mu(U)} - (\cancel{\mu(U)} - \varepsilon) = \varepsilon$$

By outer reg again, $\exists$ open $V \supseteq U \setminus E$ s.t. $\mu(V) < \varepsilon$.

Consider $K' = K \setminus V = K \cap \overset{\text{closed}}{V^c}$. K' is then cpct and clearly $K' \subseteq K$, and since $U \setminus E \subseteq V$, we also have $K' \subseteq E$.

$$\mu(K') = \mu(K) - \mu(V) > \mu(U) - \varepsilon - \varepsilon$$
$$\geq \mu(E) - 2\varepsilon.$$

Since ε arbitrary, this means that μ is is inner reg. on any $E \in \mathcal{B}_X$ w/ $\mu(E) < \infty$.

For general σ-finite E, see Folland.

$\blacksquare$

Some Approximation Results.

Thm 1 For any Radon measure μ, $\mathcal{C}_c(X)$ is dense in $L^p(X,\mu)$ for $1 \le p < \infty$.

Rem. Clearly, $\mathcal{C}_c(X)$ cannot be closed in L^∞, since the closure of $\mathcal{C}_c(X)$ under $\|\cdot\|_u$ is $\mathcal{C}_0(X) \subseteq \mathcal{C}(X)$.

Pf. By Prop 6.7, the simple fcns in L^p are dense in L^p. Suffices then to show (by linearity) that for any Borel set E w/ $\mu(E) < \infty$ and any $\varepsilon > 0$ $\exists f \in \mathcal{C}_c(X)$ s.t. $\|f - \chi_E\|_p < (2\varepsilon)^{1/p}$

By Lemma 1 ($\Rightarrow$ inner reg. on E), $\exists$ cpct $K \subseteq E$ s.t. $\mu(K) > \mu(E) - \varepsilon$

By outer reg., $\exists$ $E \subseteq U$ open s.t. $\mu(E) > \mu(U) - \varepsilon$. By Urysohn, $\exists f \in \mathcal{C}_c(X)$, $f = 1$ on K, $f \prec U$

We then have:

$$|f-\chi_E| \le |\chi_U - \chi_K| = |\chi_{U\setminus K}|$$

$$\|f-\chi_E\|_p^p \le \int |\chi_{U\setminus K}|^p = \mu(U\setminus K) =$$

$$\mu(U) - \mu(K) \le \mu(E) + \varepsilon - (\mu(E) - \varepsilon)$$
$$= 2\varepsilon.$$

$\Rightarrow \|f-\chi_E\|_p \le (2\varepsilon)^{1/p}$ as desired. ☒

<u>Lusin's Thm</u>. Let μ be Radon meas. on X. If $f: X \to \mathbb{C}$ is meas. and $f=0$ outside a set A w/ $\mu(A)<\infty$, then $\forall \varepsilon>0$ $\exists$ $E \subseteq X$ w/ $\mu(E)<\varepsilon$, and $\varphi \in C_c(X)$ s.t. $f=\varphi$ on $X\setminus E$, and $\|\varphi\|_u \le \|f\|_u$ (where $\|f\|_u = \infty$ is allowed and of course in this case $\|\varphi\|_u \le \|f\|_u = \infty$ is trivially satisfied).

We shall rely on <u>Egoroff's Thm</u> (in Ch. 2.4): If $\mu(X) < \infty$, $\{f_j\}_{j=1}^{\infty}$, f are meas. fcns $X \to \mathbb{C}$ s.t. $f_j \to f$ a.e. as $j \to \infty$, then $\forall \varepsilon > 0$ $\exists E \subseteq X$ w/ $\mu(E) < \varepsilon$ and $f_j \to f$ uniformly on $X \setminus E$. + <u>Tietze Extension Thm</u>.

<u>Pf of Lusin</u>. We shall do the case where f is bdd, i.e. $\|f\|_u \leq M$ for some M. Since $f = 0$ outside A, $\mu(A) < \infty$, we have $f \in L^1(X, \mu)$. By Thm 2, $\exists$ seq. $\{\varphi_j\}$ in $C_c(X)$ s.t. $\varphi_j \to f$ in L^1.

By previous result (Cor 2.32), there is subseq. $\{\varphi_{j_k}\}$ s.t. $\varphi_{j_k} \to f$ a.e. (μ), and by Egoroff (applied to measure space (A, μ)), $\exists E' \subseteq A$ s.t. $\varphi_j \to f$ unif. on $A \setminus E'$ and $\mu(E') < \varepsilon$.

Since $\mu(A \setminus E') < \infty$ (in particular, σ-finite) by inner reg. (Lemma 1) of $A \setminus E'$, $\exists$ cpct $K \subseteq A \setminus E'$ s.t. $\mu(A \setminus E') < \mu(K) + \varepsilon$. By outer reg., also $\exists$ open $U \supseteq A$ s.t. $\mu(U) < \mu(A) + \varepsilon$.

Since $K \subseteq A \setminus E'$, $\varphi_{j_k} \to f$ unif. on K, and hence f is continuous on K. Applying Tietze's extension to $f|_K$ in measure space (U, μ), we obtain $\varphi \in C_c(U)$ s.t. $\varphi|_K = f|_K$. We let $E = U \setminus K \Rightarrow \varphi = f$ on $X \setminus E$ (since both φ, f vanish outside U) and

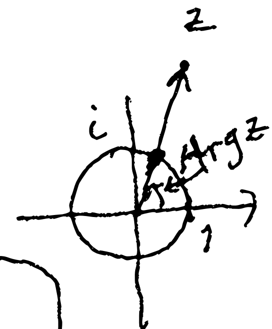
$$\mu(E) = \mu(U) - \mu(K) < \mu(A) + \varepsilon - (\mu(A) - \mu(E') - \varepsilon) = \mu(E') + 2\varepsilon < 3\varepsilon.$$

This proves 1st part of Lusin (for bdd f).

To complete the proof (for bdd f), we need to show that we can find φ s.t. $\|\varphi\|_u \le \|f\|_u$.

Define in $\mathbb{C}$, $g \in \mathcal{C}(\mathbb{C})$ by

$$g(z) = \begin{cases} z, & |z| \le \|f\|_u \\ \|f\|_u e^{i \operatorname{Arg} z}, & |z| > \|f\|_u \end{cases}$$



Continuity is clear. Let φ be any $\mathcal{C}_c(X)$ given by 1st part of Thm.; i.e. for $\varepsilon > 0$ let $\varphi \in \mathcal{C}_c(X)$, $E \subseteq X$ s.t. $\mu(E) < \varepsilon$ and $\varphi = f$ on $X \setminus E$. We claim $\tilde{\varphi} = g \circ \varphi$ satisfies the desired prop's.

Clearly, $\tilde{\varphi} \in \mathcal{C}_c(X)$ and, by def of g & φ, $\tilde{\varphi} = f$ on $X \setminus E$. For $x \in E$, either

$|\varphi(x)| \le \|f\|_u \Rightarrow |\tilde{\varphi}(x)| = |\varphi(x)| \le \|f\|_u$

or $|\varphi(x)| > \|f\|_u \Rightarrow |\tilde{\varphi}(x)| = |g(\varphi(x))| = \|f\|_u$.

This completes pf of Lusin in the case $\|f\|_u < \infty$. For unbdd f, see Folland. $\blacksquare$